Kuwait University	**Math 101**	**Date:**	**November 8, 2010**
Department of Mathematics	**First Exam**	**Duration:**	**90 minutes**

Calculators, cellular phones and all other mobile communication equipments are not allowed

Answer the following questions:

1. Find the vertical and horizontal asymptotes, if any, for the graph of
$$f(x) = \frac{x\,|x|}{x^2 - x}.$$ (4 pts.)

2. Find the x-coordinates of the points at which the function f is discontinuous, where
$$f(x) = \begin{cases} \dfrac{x^2 - x}{x^3} & \text{, if } x < -1, \\ 2 & \text{, if } -1 \leq x < 1, \\ 3 & \text{, if } x = 1, \\ x + 1 & \text{, if } 1 < x \leq 2, \\ \dfrac{x^3 - 1}{x^2 - 3x + 2} & \text{, if } x > 2. \end{cases}$$
Classify the types of discontinuity of f as removable, jump, or infinite. (4 pts.)

3. Show that the equation $2x\sin x + x + 1 = 0$ has a real solution. (3 pts.)

4. Find the x-coordinate of the points at which $f(x) = x^{\frac{2}{3}} - x^{\frac{5}{3}}$ has
   (a) a vertical tangent line,
   (b) a horizontal tangent line. (4 pts.)

5. Use the definition of the derivative to show that f is differentiable at -1, where
$$f(x) = \begin{cases} -2x - 1 & \text{, if } x \leq -1, \\ x^2 & \text{, if } x > -1. \end{cases}$$ (3 pts.)

6. Find $f'(x)$, where $f(x) = \sin^2\left(1 + \sqrt{x}\right)$. (3 pts.)

7. Evaluate the following limits, if they exist:
   (a) $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + x} - x\right)$. (2 pts.)
   (b) $\lim\limits_{x \to 0} \dfrac{x}{2 + \sin\left(\frac{1}{x}\right)}$. (2 pts.)

Kuwait University	Math 101	Date:	November 8, 2010
Department of Mathematics	First Exam		Answer Key

1. $\lim\limits_{x\to 1^{\pm}} f(x) = \lim\limits_{x\to 1^{\pm}} \dfrac{x\,|x|}{x(x-1)} = \pm\infty \implies \boxed{x=1 \text{ is V.A}}$

$\lim\limits_{x\to \pm\infty} f(x) = \lim\limits_{x\to \pm\infty} \dfrac{\pm x^2}{x^2\left(1-\frac{1}{x}\right)} = \pm 1 \implies \boxed{y=1 \text{ and } y=-1 \text{ are H.A.}}$

2. $\lim\limits_{x\to -1^{-}} f(x) = \lim\limits_{x\to -1^{-}} \dfrac{x^2-x}{x^3} = -2, \ \lim\limits_{x\to -1^{+}} f(x) = \lim\limits_{x\to -1^{+}} 2 = 2.$

f has a *jump discontinuity* at $x=-1$.

$f(1) = 3, \ \lim\limits_{x\to 1^{+}} f(x) = \lim\limits_{x\to 1^{+}} (x+1) = 2, \ \lim\limits_{x\to 1^{-}} f(x) = \lim\limits_{x\to 1^{-}} 2 = 2.$

f has a *removable discontinuity* at $x=1$.

$f(2) = 3, \ \lim\limits_{x\to 2^{-}} f(x) = \lim\limits_{x\to 2^{-}} (x+1) = 3, \ \lim\limits_{x\to 2^{+}} f(x) = \lim\limits_{x\to 2^{+}} \frac{x^3-1}{(x-1)(x-2)} = \infty.$

f has an *infinite discontinuity* at $x=2$.

3. Let $f(x) = 2x\sin x + x + 1$. f is *continuous* on $[-\pi, 0]$, and $f(-\pi) = -\pi + 1 < 0$ & $f(0) = 1 > 0$. From The Intermediate Value Theorem, there is at least one $c \in (-\pi, 0)$ such that $f(c) = 0$. Thus, c is a real solution for the equation.

4. $f'(x) = \frac{2}{3\sqrt[3]{x}} - \frac{5}{3}x^{\frac{2}{3}} = \frac{2-5x}{3\sqrt[3]{x}}$

   (a) f is continuous at $x=0$, $\lim\limits_{x\longrightarrow 0^{\pm}} f'(x) = \pm\infty$. The graph of f has a vertical tangent at $x=0$.

   (b) $f'\left(\frac{2}{5}\right) = 0$. Thus, f has a horizontal tangent at $x = \frac{2}{5}$.

5. L.H.D. at $(x=-1) = \lim\limits_{x\longrightarrow -1^{-}} \frac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\longrightarrow -1^{-}} \frac{(-2x-1)-1}{x+1} = -2$

R.H.D. at $(x=-1) = \lim\limits_{x\longrightarrow -1^{+}} \frac{f(x)-f(-1)}{x-(-1)} = \lim\limits_{x\longrightarrow -1^{+}} \frac{x^2-1}{x+1} = -2$

$f'(-1) = \lim\limits_{x\longrightarrow -1} \frac{f(x)-f(-1)}{x-(-1)} = -2 \in \mathbb{R}$. Therefore, f is differentiable at $x=-1$.

6. $f'(x) = 2\sin\left(\sqrt{x}+1\right)\cos\left(\sqrt{x}+1\right)\dfrac{1}{2\sqrt{x}}.$

7. (a) $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x} - x\right) = \lim\limits_{x\to\infty} \left(\sqrt{x^2+x} - x\right) \times \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim\limits_{x\to\infty} \frac{\left(x^2+x\right)-x^2}{\sqrt{x^2\left(1+\frac{1}{x}\right)}+x}$

$= \lim\limits_{x\to\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x}}+x} = \lim\limits_{x\to\infty} \frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)} = \boxed{\frac{1}{2}}$

   (b) For $x \neq 0$, $-1 \le \sin\frac{1}{x} \le 1 \implies 1 \le 2 + \sin\left(\frac{1}{x}\right) \le 3 \implies \boxed{\frac{1}{3} \le \frac{1}{2+\sin\left(\frac{1}{x}\right)} \le 1}$

   (I) If $x > 0$, $\frac{x}{3} \le \frac{x}{2+\sin\left(\frac{1}{x}\right)} \le x$. Since $\lim\limits_{x\to 0^{+}} \frac{x}{3} = 0 = \lim\limits_{x\to 0^{+}} x$, then from the Squeeze Theorem $\lim\limits_{x\to 0^{+}} \frac{x}{2+\sin\left(\frac{1}{x}\right)} = 0$.

   (II) If $x < 0$, $\frac{x}{3} \ge \frac{x}{2+\sin\left(\frac{1}{x}\right)} \ge x$. Since $\lim\limits_{x\to 0^{-}} \frac{x}{3} = 0 = \lim\limits_{x\to 0^{-}} x$, then from the Squeeze Theorem $\lim\limits_{x\to 0^{-}} \frac{x}{2+\sin\left(\frac{1}{x}\right)} = 0$.

   Therefore, $\lim\limits_{x\to 0} \frac{x}{2+\sin\left(\frac{1}{x}\right)} = 0$.